

On Schwinger's formula for pair production

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Abstract

We present some comments on Schwingers's calculation of electron-positron production in a prescribed constant electric field. The range of validity of $2Im\mathcal{L}^{(1)}(E)$ is discussed thoroughly and limiting cases are provided.

1 Number of electron-positron pairs produced in a uniform electric field

Start with Schwinger's expression for $2Im\mathcal{L}^{(1)}(E)$ [1]:

$$2Im\mathcal{L}^{(1)} = 2 \underbrace{\frac{(eE)^2}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{\pi n}{\beta}}}_{= Li_2(e^{-\frac{\pi}{\beta}}), \text{ Euler's dilogarithm}}, \quad \beta = \frac{eE}{m^2} \quad (1.1)$$

$\uparrow^z \mathbf{E} = const.$

$\hbar = c = 1, \quad V = L^3$

In general for spin $s = \frac{1}{2}$ and $s = 0$:

$$\begin{aligned} 2Im\mathcal{L}^{(1)}(E) &= (2s+1) \frac{(eE)^2}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{(\pm 1)^{n+1}}{n^2} e^{-n \frac{\pi}{\beta}} \\ &= \pm (2s+1) \frac{(eE)^2}{(2\pi)^3} Li_2(\pm e^{-\frac{\pi}{\beta}}). \end{aligned} \quad (1.2)$$

Continuous phase space integration, how to count states:

$$\int d^3\mathbf{p} \frac{V}{(2\pi)^3} \cdot / \cdot = \int_{-\infty}^{+\infty} \frac{L}{2\pi} dp_1 \int_{-\infty}^{+\infty} \frac{L}{2\pi} dp_2 \int_{-\infty}^{+\infty} \frac{L}{2\pi} dp_3 \cdot / \cdot \quad (1.3)$$

$$t = \frac{p_3}{eE} \text{ in const. } E\text{-field: } dp_3 = eE dt, \quad 0 \leq t \leq T \text{ or } \int_{-\infty}^{+\infty} dp_3 \rightarrow eET.$$

Replace one factor (eE) in (1.1) by $\frac{1}{T} \int_{-\infty}^{+\infty} dp_3$. Then

$$Im\mathcal{L}^{(1)}(E)T = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dp_3 \frac{(eE)}{(2\pi)} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{\pi n}{\beta}}. \quad (1.4)$$

Rewrite the sum in (1.4):

$$\begin{aligned} & \frac{eE}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{\pi}{\beta} n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{\pi}{\beta} n} \frac{1}{2} \frac{1}{\pi \frac{n}{eE}}, \quad \frac{1}{a} = \int_0^{\infty} dx e^{-ax}, \quad a = \pi \frac{n}{eE} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{\pi}{\beta} n} \frac{1}{2} \int_0^{\infty} d(p_{\perp}^2) e^{-\pi \frac{n}{eE} p_{\perp}^2}, \quad dp_{\perp}^2 = 2p_{\perp} dp_{\perp}, \quad \beta = \frac{eE}{m^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} p_{\perp} dp_{\perp} \exp \left\{ -\pi \frac{m^2 + p_{\perp}^2}{eE} n \right\} \\ &= \int_0^{\infty} dp_{\perp} p_{\perp} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left\{ -\pi \frac{m^2 + p_{\perp}^2}{eE} n \right\} \Rightarrow \end{aligned}$$

$$\text{Use } \ln(1-x) = - \sum_{n=1}^{\infty} \frac{1}{n} x^n, \quad |x| < 1, \quad x = \exp \left\{ -\pi \frac{m^2 + p_{\perp}^2}{eE} \right\}$$

$$\Rightarrow - \int_0^{\infty} dp_{\perp} p_{\perp} \ln(1 - e^{-\pi \lambda_p}) \Rightarrow, \quad \lambda_p = \frac{m^2 + p_{\perp}^2}{eE} \quad (1.5)$$

$$e^{-\pi \lambda_p} = e^{-\pi \frac{m^2 + p_{\perp}^2}{eE}} = \bar{n}_p. \quad (1.6)$$

This important expression - also probability for tunneling - relates the imaginary part of the Lagrangian of the field to the mean number \bar{n}_p of electron-positron

pairs produced by the field in the state with given momentum and spin projection. \bar{n}_p is degenerate with respect to spin (two) and momentum p_3 with $\frac{L_3 \Delta p_3}{2\pi\hbar}$ with $\Delta p_3 = eET$.

So we can continue to write (1.5):

$$\Rightarrow - \int_0^\infty dp_\perp p_\perp \ln(1 - \bar{n}_p), \quad \int_0^\infty dp_\perp 2\pi p_\perp \cdot / \cdot = \int_{-\infty}^\infty dp_1 dp_2 \cdot / \cdot$$

Here then is the relation between $Im\mathcal{L}^{(1)}(E)$ and \bar{n}_p (insert \hbar and $V = L^3$):

$$\frac{2}{\hbar} Im\mathcal{L}^{(1)} VT = -2 \int d^3\mathbf{p} \frac{V}{(2\pi)^3} \ln(1 - \bar{n}_p) \quad (1.7)$$

$$\bar{n}_p = \exp \left\{ -\pi \frac{m^2 + p_\perp^2}{eE} \right\}. \quad (1.8)$$

This is Nikishov's virial representation of the imaginary part of $\mathcal{L}^{(1)}(E)$ [2].

With the aid of (1.8) let us prove Nikishov's result for the mean number of pairs in four-volume VT by counting states:

$$\begin{aligned} \bar{n} &= 2 \int_{-\infty}^{+\infty} dp_1 \frac{L}{(2\pi)} \int_{-\infty}^{+\infty} dp_2 \frac{L}{(2\pi)} \int_{-\infty}^{eEL} dp_3 \frac{T}{(2\pi)} \bar{n}_p \Rightarrow \\ &= 2 \frac{L^2}{(2\pi)^2} 2\pi \int_0^\infty p_\perp dp_\perp \frac{T}{2\pi} eEL e^{-\pi\lambda_p} \\ &= \frac{(eE)}{(2\pi)^2} VT \int_0^\infty dp_\perp^2 e^{-\pi \frac{p_\perp^2}{eE}} e^{-\frac{\pi}{\beta}} = \frac{eE}{(2\pi)^2} VT \frac{1}{\pi \frac{1}{eE}} e^{-\frac{\pi}{\beta}}. \end{aligned}$$

Therefore

$$\bar{n} = 2 \frac{(eE)^2}{(2\pi)^3} VT e^{-\frac{\pi}{\beta}}, \quad \beta = \frac{eE}{m^2}, \quad (1.9)$$

which is Nikishov's result for the mean number of pairs produced in volume $V = L^3$ during time T . For Bose particles the factor 2 is suppressed.

Introducing

$$\xi = \exp \left\{ -\frac{\pi m^2}{eE} \right\}, \quad \gamma = VT \frac{(eE)^2}{4\pi^3} \quad (1.10)$$

we also can write

$$\bar{n} = \gamma \xi. \quad (1.11)$$

Formula (1.9) is an approximation of the following expression for $n = 1$:

$$\int d^3p \frac{V}{(2\pi)^3} \sum_{n=1}^{\infty} \bar{n}_p^n. \quad (1.12)$$

Furthermore the well-known vacuum persistence probability $|\langle O_+ | O_- \rangle|^2$ is given by

$$\begin{aligned} |\langle O_+ | O_- \rangle|^2 &= p_0 = \prod_{s,p} (1 - e^{-\pi \lambda_p}) \\ &= \prod_{s,p} \left(1 - e^{-\pi \frac{m^2 + p_\perp^2}{eE}} \right) = e^{-2Im\mathcal{L}^{(1)}(E)VT} \\ &= \exp \left\{ -VT \frac{(eE)^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n \frac{\pi m^2}{eE}} \right\} \\ &= \exp \{ -\gamma Li_2(\xi) \}. \end{aligned} \quad (1.13)$$

Schwinger writes in his brilliant article [1] as well as in his 2nd volume on “sources, particles and fields”: “We recognize in $2Im\mathcal{L}^{(1)}$ a measure in the probability, per unit time and unit spatial volume, that an electron-positron pair has been created.” This statement is only true for very weak fields ($eE \ll m^2$) in which case the contributions of the $n = 2, 3, \dots$ terms in the sum of (1.1) can be neglected:

$$2Im\mathcal{L}^{(1)}(eE \ll m^2) = 2 \frac{(eE)^2}{(2\pi)^3} e^{-\frac{\pi}{\beta}} \equiv \frac{\bar{n}}{VT} = \frac{\gamma\xi}{VT} \simeq \frac{\gamma L}{VT}; \quad L = -\ln(1 - \xi),$$

which is identical to Nikishov’s result. Here, in order to save at least part of Schwinger’s statement, we are being a bit casual since \bar{n} is not a probability but an average number. To be more specific let us start with (1.12):

$$\begin{aligned} & 2 \int d^3p \frac{V}{(2\pi)^3} \sum_{n=1}^{\infty} (\bar{n}_p)^n, \quad \int dp_3 = TeE \\ &= V \frac{TeE}{4\pi^3} \int_{-\infty}^{+\infty} dp_1 \int_{-\infty}^{+\infty} dp_2 \sum_{n=1}^{\infty} (\bar{n}_p)^n, \quad \int dp_1 \int dp_2 = 2\pi \int_0^{\infty} dp_\perp p_\perp \\ &= VT \frac{(eE)}{2\pi^2} \int_0^{\infty} p_\perp dp_\perp \sum_{n=1}^{\infty} e^{-\pi \lambda_p n}, \quad \bar{n}_p = e^{-\pi \lambda_p}, \quad \lambda_p = \frac{m^2 + p_\perp^2}{eE}, \quad \beta = \frac{eE}{m^2} \\ &= VT \frac{(eE)}{2\pi^2} \sum_{n=1}^{\infty} e^{-\frac{\pi}{\beta} n} \frac{1}{2} \int_0^{\infty} dp_\perp^2 e^{-\pi \frac{n}{eE} p_\perp^2} \\ &= VT \frac{(eE)}{2\pi^2} \sum_{n=1}^{\infty} e^{-\frac{\pi}{\beta} n} \frac{1}{2 \frac{\pi n}{eE}} = VT \frac{(eE)^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{\pi m^2}{eE} n} \\ &= VT \frac{(eE)^2}{4\pi^3} \left[-\ln(1 - e^{-\pi \frac{m^2}{eE}}) \right] =: \frac{p_1}{p_0} = \gamma L. \end{aligned}$$

Finally

$$\begin{aligned} p_1 &= VT \frac{(eE)^2}{4\pi^3} \left[-\ln \left(1 - e^{-\pi \frac{m^2}{eE}} \right) \right] \cdot \exp \left\{ -VT \frac{(eE)^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n \frac{\pi m^2}{eE}} \right\} \\ p_1 &= \gamma L p_0, \quad L = -\ln(1 - \xi), \quad p_0 = \exp \{ -\gamma Li_2(\xi) \}. \end{aligned} \quad (1.14)$$

So far we have

$$\begin{aligned} p_0 &= \exp \{ -\gamma Li_2(\xi) \}, & \text{Schwinger's vacuum persistence probability} \\ p_1 &= \gamma L p_0, \\ \bar{n} &= \gamma \xi. \end{aligned}$$

Let's denote by $\alpha = (\mathbf{p}, s)$ the quantum numbers of the electron states. Then we can write our vacuum-to-vacuum probability as

$$p_0 = \prod_{\alpha} (1 - \bar{n}_{\alpha}). \quad (1.15)$$

In an electric field any number of pairs can be produced, so the probability that there are $n = 0, 1, 2, \dots$ electron-positron pairs shows up in the series

$$\begin{aligned} \sum_{n=0}^{\infty} p_n &= \prod_{\alpha} (1 - \bar{n}_{\alpha}) + \sum_{\alpha} \bar{n}_{\alpha} \prod_{\beta \neq \alpha} (1 - \bar{n}_{\beta}) \\ &+ \frac{1}{2!} \sum_{\alpha \neq \beta} \bar{n}_{\alpha} \bar{n}_{\beta} \prod_{\gamma \neq \alpha, \beta} (1 - \bar{n}_{\gamma}) + \dots = 1. \end{aligned} \quad (1.16)$$

In our case, each of the quantities p_n describes the probability for the number $n = 0, 1, 2, \dots$ electron-positron pairs in four-volume. The first terms were calculated above:

$$\begin{aligned} p_0 &= \exp \{ -\gamma Li_2(\xi) \} \\ p_1 &= \gamma L p_0. \end{aligned} \quad (1.17)$$

The next and followers for the numbers n were calculated by Krivoruchenko:

$$p_2 = \frac{\gamma}{2} \left(\gamma L^2 + L - \frac{\xi}{1 - \xi} \right) p_0 \quad \text{etc.} \quad (1.18)$$

It is highly interesting to follow Krivoruchenko's paper [3] and find out that in electric fields of supercritical strength $|eE| > \frac{\pi m^2}{ln2}$, the unitary condition $\sum_{n=0}^{\infty} p_n = 1$, changes into an asymptotic divergence, i.e. the positive definiteness of the probability is violated. This divergence indicates a failure of the continuum limit approximation, i.e. by the replacement of the discrete sum by the integral over the phase space:

$$\sum_{\alpha} ./. \rightarrow 2 \int \frac{V}{(2\pi)^3} d^3 \mathbf{p} \dots = VT |eE| \int \frac{2d^2 p_{\perp}}{(2\pi)^3} \dots \quad (1.19)$$

2 Schwinger's formula for $Im\mathcal{L}^{(1)}(E)$ the long way, i.e., without using the residue theorem.

Take the formula (5.27) or equivalently (6.33) of the “Lecture Notes 220” on “Effective Lagrangians in QED” by Dittrich and Reuter [4, 5]:

$$\begin{aligned}\mathcal{L}^{(1)}(B) = & -\frac{1}{32\pi^2} \left\{ (2m^4 - 4m^2(eB) + \frac{4}{3}(eB)^2) \left[1 + \ln\left(\frac{m^2}{2eB}\right) \right] \right. \\ & \left. + 4m^2(eB) - 3m^4 - (4eB)^2 \zeta'\left(-1, \frac{m}{2eB}\right) \right\}. \quad (2.1)\end{aligned}$$

This can also be written in the form

$$\begin{aligned}\mathcal{L}^{(1)}(B) = & -\frac{1}{32\pi^2} \left\{ -3m^4 + 4(eB)^2 \left(\frac{1}{3} - 4\zeta'(-1) \right) + 4m^2(eB) \ln 2\pi - 1 \right. \\ & - 2m^4 \ln \frac{2eB}{m^2} - 4m^2(eB) \ln \frac{2eB}{m^2} - \frac{4}{3}(eB)^2 \ln \frac{2eB}{m^2} \\ & \left. - 16(eB)^2 \int_1^{1+\frac{m^2}{2eB}} dx \ln \Gamma(x) \right\}. \quad (2.2)\end{aligned}$$

Introducing the critical field strength $B_{er} = \frac{m^2}{e}$ and measuring the magnetic field in this unit, we can rewrite the last expression as

$$\begin{aligned}\mathcal{L}^{(1)}(B) = & \frac{\alpha}{2\pi} \left\{ \frac{3}{4} - B(\ln(2\pi) - 1) - B^2 \left(\frac{1}{3} - 4\zeta'(-1) \right) \right. \\ & \left. + \left(\frac{1}{2} + B + \frac{1}{3}B^2 \right) \ln(2B) + 4B^2 \int_1^{1+\frac{1}{2B}} \ln(\Gamma(x)) dx \right\}. \quad (2.3)\end{aligned}$$

For a pure electric field the Lagrangian is likewise given by

$$\begin{aligned}\mathcal{L}^{(1)}(E) = & -\frac{1}{32\pi^2} \left\{ \left(2m^4 + 4im^2eE - \frac{4}{3}e^2E^2 \right) \left(\ln\left(i\frac{m^2}{2eE}\right) + 1 \right) \right. \\ & \left. - 3m^4 - 4im^2eE + 16e^2E^2 \zeta'\left(-1, i\frac{m^2}{2eE}\right) \right\}. \quad (2.4)\end{aligned}$$

It takes a little practice to separate this formula into its real and imaginary part:

$$\begin{aligned}
\mathcal{L}^{(1)}(E) &= \frac{\alpha}{2\pi} \left\{ \frac{3}{4} + \frac{1}{2} \ln(2E) - \frac{\pi}{2} E + E^2 \left(\frac{1}{3} - 4\zeta'(-1) \right) - \frac{1}{3} E^2 \ln(2E) \right. \\
&\quad \left. + 4E^2 \int_0^{1/2E} \operatorname{Im} \ln(\Gamma(1+y)) dy \right\} \\
&\quad + i \frac{\alpha}{2\pi} \left\{ -\frac{\pi}{4} - E \ln(2E) + E(\ln(2\pi) - 1) \right. \\
&\quad \left. + \frac{\pi}{6} E^2 - 4E^2 \int_0^{1/2E} \operatorname{Re} \ln(\Gamma(1+iy)) dy \right\}. \tag{2.5}
\end{aligned}$$

With the aid of the relation (Gradshteyn / Ryzhik)

$$\begin{aligned}
\operatorname{Re} \ln(\Gamma(1+iy)) &= \ln |\Gamma(1+iy)| = \frac{1}{2} \ln [\Gamma(1+iy)\Gamma(1-iy)] \\
&= -\frac{1}{2} \ln \frac{\sinh(\pi y)}{\pi y}
\end{aligned}$$

and an integration by parts

$$\int_0^{1/2E} 1 \cdot \ln \frac{\sinh(\pi y)}{\pi y} dy = \frac{1}{2E} \ln \left[\frac{2E}{\pi} \sinh \frac{\pi}{2E} \right] - \int_0^{1/2E} [\pi y \coth(\pi y) - 1] dy,$$

we obtain

$$\begin{aligned}
\operatorname{Im} \mathcal{L}^{(1)}(E) &= \frac{\alpha}{2\pi} \left\{ -\frac{\pi}{4} + E \ln 2 + \frac{\pi}{6} E^2 + E \ln \left(\sinh \frac{\pi}{2E} \right) \right. \\
&\quad \left. - 2E^2 \int_0^{1/2E} \pi y \coth(\pi y) dy \right\}. \tag{2.6}
\end{aligned}$$

Let's change the variable $x = \pi y$ and evaluate the integral on the right-hand side with the use of [Wolfram Mathematica online integrator]

$$\begin{aligned}
\int x \coth(x) dx &= \frac{1}{2} (x(x + 2 \ln(1 - e^{-2x})) - Li_2(e^{-2x})) \\
&= \frac{1}{2} (x(x + 2 \ln(2e^{-x} \sinh(x)) - Li_2(e^{-2x})) . \tag{2.7}
\end{aligned}$$

With integration limits we arrive at

$$\begin{aligned}
\frac{1}{\pi} \int_0^{\frac{\pi}{2}E} x \coth(x) dx &= \frac{1}{2\pi} \left[\frac{\pi}{2E} \left(\frac{\pi}{2E} + 2 \ln 2 - \frac{\pi}{E} + 2 \ln \left(\sinh \left(\frac{\pi}{2E} \right) \right) \right) \right. \\
&\quad \left. - Li_2 \left(e^{-\frac{\pi}{E}} \right) + \frac{\pi^2}{6} \right]. \tag{2.8}
\end{aligned}$$

We substitute this in our last expression for $Im\mathcal{L}^{(1)}(E)$ and obtain

$$\begin{aligned}
Im\mathcal{L}^{(1)}(E) &= \frac{\alpha}{2\pi} \left\{ -\frac{\pi}{4} + E \ln 2 + \frac{\pi}{6} E^2 + E \ln \left(\sinh \left(\frac{\pi}{2E} \right) \right) \right. \\
&\quad \left. + \frac{\pi}{4} - E \ln 2 - E \ln \left(\sinh \left(\frac{\pi}{2E} \right) \right) + \frac{E^2}{\pi} Li_2 \left(e^{-\frac{\pi}{E}} \right) - \frac{\pi}{6} E^2 \right\} \\
&= \frac{\alpha}{2\pi} \frac{E^2}{\pi} Li_2 \left(e^{-\frac{\pi}{E}} \right) \\
\left(Li_2 \left(e^{-\frac{\pi}{E}} \right) \right) &= \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{\pi}{E} n}; \\
\text{in units of } E_{cr} &= \frac{m^2}{e} \quad : \quad \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{m^2 \pi}{e E} n}. \tag{2.9}
\end{aligned}$$

Finally we obtain J.S.'s famous result ($\alpha = \frac{e^2}{4\pi}$):

$$Im\mathcal{L}^{(1)}(E) = \frac{\alpha}{2\pi^2} E^2 \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{m^2 \pi}{e E} n}. \tag{2.10}$$

At last we might add the result for the real part [6]:

$$\begin{aligned}
Re\mathcal{L}^{(1)}(E) &= -\frac{e^2 E^2}{4\pi^4} \left(C + \ln \frac{\pi m^2}{e E} \right) \sum_{n=1}^{\infty} \frac{1}{n^2} \cosh \left\{ n \frac{\pi m^2}{e E} \right\} \\
&\quad - \frac{e^2 E^2}{4\pi^4} \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \cosh \left\{ n \frac{\pi m^2}{e E} \right\}, \tag{2.11}
\end{aligned}$$

where C is the Euler-Mascheroni constant.

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 Replace in this reference in (3.25) ζ' by $-\frac{1}{2}\zeta'$ and $\ln \frac{m^2}{eH}$ by $\ln \frac{m^2}{2eH}$.
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